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Modified systems found by symmetry reduction on the cotangent bundle of a loop group

Ian Marshall [1]

Department of Mathematics, Leeds University, Leeds LS2 9JT, UK
e-mail: amt6im@amsta.leeds.ac.uk
Cimat, Apdo. Postal 402, Guanajuato, Gto. 36000, Mexico

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Abstract

The cotangent bundle $T^*\tilde{G}$ of a loop group is considered, not with the canonical symplectic structure, but with a deformation of it engendered by the nontrivial cocycle $\Sigma : \tilde{G} \to \tilde{\mathfrak{g}}^*$. Three symplectic actions on $T^*\tilde{G}$ are then considered; the left- and right-actions of $\tilde{G}$ and that of the Diffeomorphism group $Diff(S^1)$. Several examples of systems "of modified type" are shown to arise by combining the moment maps for these actions, in conjunction with the r-matrix construction.

An abstraction of this idea is discussed and it is shown that the Clebsch integrable case of the motion of a rigid body in an ideal fluid is an example of a system which can be described via precisely the same geometric approach.

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1. Introduction

Let $\mathcal{L}$ be the Lie algebra $\mathfrak{d} \ltimes \tilde{\mathfrak{g}}$ where $\mathfrak{d} = Vect(S^1)$ and $\tilde{\mathfrak{g}} = C^\infty(S^1, gl_n)$. $\mathcal{L}$ is the same as $\{\phi\mathbf{I}\partial + \mathbf{v} \mid \phi \in C^\infty(S^1, \mathbb{R}),\ \mathbf{v} \in C^\infty(S^1, gl_n)\}$, with the Lie bracket given by commutation of operators ($\mathbf{I}$ is the identity matrix). In Ref. [2] the central extension $\hat{\mathcal{L}}$ of $\mathcal{L}$ was considered: the r-matrix construction was applied to the loop algebra $\ell(\hat{\mathcal{L}}) = \hat{\mathcal{L}} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ of $\hat{\mathcal{L}}$ and it was found that many results of Kupershmidt [4] could be seen as special cases. Several results of Antonowicz and Fordy [3] were

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also reproduced; this corresponds to the scalar case $n = 1$, and was explained in Ref. [1] originally. The results both of Kupershmidt and of Antonowicz and Fordy involved finding generalised Miura maps. It was shown in Ref. [2] that these maps are all "encoded" in an equivariant Poisson mapping $\mathbf{m} : \hat{\mathcal{L}}^* \to VIR^*$ from $\hat{\mathcal{L}}^*$ to the dual of the Virasoro algebra. An immediate consequence, sufficient to establish the connection with Refs. [3] and [4], is that the set of coadjoint invariants on VIR^* is pulled back by $\mathbf{m}$ to give coadjoint invariants on $\hat{\mathcal{L}}^*$. In Ref. [5] it was shown how this approach can be used to construct a modification of Ito's equation; this was interesting because the approach of Antonowicz and Fordy precluded the existence of such a modification.

Remark. There is a well-known construction for obtaining integrable systems for any Lie algebra, known as the AKS-construction (for Adler–Kostant–Symes) or the r-matrix approach. It means that one essentially proceeds by saying

"Think of a Lie algebra $\mathfrak{g}$. Find the set $I(\mathfrak{g}^*)$ of Ad*-invariants on the dual."

One constructs $\ell(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ and then carries out some standard computations. A brief description of this construction is given in the Appendix for completeness. We do not concern ourselves here with it other than to use it to compute the flow in Section 5. A thorough account of the technique can be found in Ref. [9].

The present paper has two objectives.

1. The primary objective is the presentation of the results of Ref. [2] from a geometric point of view. It becomes rather obvious how some of the mysterious formulae in Ref. [2] (and hence also those in Refs. [3] and [4]) come about. In particular the mapping $\mathbf{m}$ is given an interpretation in terms of a dual pair of Poisson maps. Also we get a simple proof that the set $I(\hat{\mathcal{L}}^*)$ of Ad*-invariant functions on $\hat{\mathcal{L}}^*$ is exactly the set $\mathbf{m}^*(I(VIR^*))$ given in Ref. [2]: there it was only recognised that $\mathbf{m}^*(I(VIR^*)) \subset I(\hat{\mathcal{L}}^*)$. ($\mathbf{m}^*$ is the pull-back mapping of functions on VIR^* to functions on $\mathcal{L}^*$.)

What will be established is the connection with the results of Ref. [2] and the following well-known picture:

Consider the cotangent lift to T^*H of the left and right actions of H on H, where H is any Lie group. The moment maps for these two actions form a dual pair; this is a reflection of the even more well-known fact that left multiplication and right multiplication commute with one another.

2. One is naturally led to ask: Are there any interesting consequences of this geometric description? The results of Ref. [2] become so much clearer, even in the simplest $n = 1$ case, that it is tempting to look for new kinds of systems by similar means. This secondary objective is left to some extent unresolved. It seems possible though that the idea suggested might lead to some interesting results.

A very important context within which to treat the Miura map seems to be that of a semidirect product Lie algebra (or group), moreover it seems to be important that this Lie algebra have a nontrivial central extension. For evidence of this one can look at several of Kupershmidt's papers, such as those in Ref. [4], as well as at Refs. [1] and [2]. A natural way of studying a semidirect product Lie group $\mathcal{A} \ltimes \mathcal{B}$ is to look at its action on $T^*\mathcal{B}$: one can lift the left and right actions of $\mathcal{B}$ on $\mathcal{B}$ as well as the action of $\mathcal{A}$ on $\mathcal{B}$ to $T^*\mathcal{B}$. The results of Ref. [2] follow almost automatically when we take $\mathcal{B}$ to be the loop group $\mathcal{B} = \tilde{G} = C^\infty(S^1, G)$, adding the central extension to the Lie algebra $\tilde{\mathfrak{g}}$, and $\mathcal{A} = Diff(S^1)$.

The "idea" suggested here then amounts to little more than to say "Look at semidirect products having central extensions". However we will show that the interesting systems so obtained are not restricted only to those of Ref. [2]; the Clebsch integrable case of a rigid body in an ideal fluid can be described in *exactly* this way with some surprising similarities to the systems of Ref. [2]. It is one of a class of systems found from the case when $\mathcal{B} = V$ is a vector space – the group product law is addition, i.e. $\mathcal{B}$ is abelian – and $\mathcal{A}$ is a certain subgroup of $GL(V)$.

Summary of Ref. [2]

Let $\mathcal{G}$ be the Lie group

$$\mathcal{G} = Diff(S^1) \ltimes \tilde{G}. \tag{1.1}$$

Here $\tilde{G} = C^\infty(S^1, G)$ with G any finite dimensional Lie group. The product rule on $\mathcal{G}$ is given by

$$((\sigma_1, g_1)(\sigma_2, g_2)) \mapsto (\sigma_1, g_1)(\sigma_2, g_2) = (\sigma_2 \circ \sigma_1, g_1 g_2 \circ \sigma_1). \tag{1.2}$$

(Note that the product defined here on the first component – $Diff(S^1)$ – is not the same as that often used. This is not important, but it conforms with the convention of Ref. [2].)

In Ref. [2], the construction of the r-matrix approach was applied to the loop algebra $\ell(\hat{\mathcal{L}})$, where $\hat{\mathcal{L}}$ is the extended Lie algebra of $\mathcal{G}$:

Assume that the Lie algebra $\mathfrak{g}$ of G has an Ad-invariant inner product $(\ ,\ )$ and use it to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Let $A \in C(\mathfrak{g})$, i.e. $[XA] = 0\ \forall X \in \mathfrak{g}$. We find that $\mathcal{G}$ has three nontrivial coadjoint 1-cocycles. The formulae for the adjoint and coadjoint actions respectively on the (3 times) centrally extended Lie algebra $\hat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$, where $\mathcal{L}$ is the Lie algebra of $\mathcal{G}$, and on its dual $\hat{\mathcal{L}}^*$, are given by

$$\begin{aligned} &\hat{\mathrm{ad}}_{(\phi_1, v_1, a_1, b_1, c_1)}(\phi_2, v_2, a_2, b_2, c_2) \\ &\quad = \Big(\phi_1\phi_2' - \phi_1'\phi_2,\ [v_1, v_2] + \phi_1 v_2' - \phi_2 v_1', \\ &\qquad\qquad \int_{S^1}(v_1, v_2'),\ \int_{S^1}(A, \phi_1'' v_2 - \phi_2'' v_1),\ \int_{S^1}\phi_1\phi_2'''\Big), \\ &(\phi_i, v_i, a_i, b_i, c_i) \in C^\infty(S^1, \mathbb{R}) \times C^\infty(S^1, \mathfrak{g}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \sim \hat{\mathcal{L}}; \end{aligned} \tag{1.3}$$

$$\hat{\mathrm{ad}}^*_{(\phi,v)}(u,\xi,e_1,e_2,e_3)$$
$$= \Big(2\phi' u + \phi u' + (\xi, v') + e_2(A, v'') + e_3\phi''',$$
$$(\phi\xi)' + [v,\xi] + e_1 v' - e_2\phi'' A, 0,0,0\Big),$$
$$(\phi, v) \in C^\infty(S^1,\mathbb{R}) \times C^\infty(S^1,\mathfrak{g}) \sim \mathcal{L},$$
$$(u,\xi,e_1,e_2,e_3) \in C^\infty(S^1,\mathbb{R}) \times C^\infty(S^1,\mathfrak{g}) \times \mathbb{R}\times\mathbb{R}\times\mathbb{R} \sim \hat{\mathcal{L}}^*; \qquad (1.4)$$

$$\hat{\mathrm{Ad}}^*_{(\sigma, g\circ\sigma)}(u,\xi,e_1,e_2,e_3)$$
$$= \Big(\sigma'^2(u + (\xi, g^{-1}g') + \tfrac{1}{2}e_1(g^{-1}g', g^{-1}g') + e_2(A,(g^{-1}g')')\circ\sigma + e_3 S(\sigma),$$
$$\sigma'(g\xi g^{-1} + e_1 g' g^{-1})\circ\sigma - e_2\frac{\sigma''}{\sigma'}A, e_1, e_2, e_3\Big),$$
$$(\sigma, g\circ\sigma) \in \mathcal{G}, (u,\xi,e_1,e_2,e_3) \in \hat{\mathcal{L}}^*. \qquad (1.5)$$

S is the Schwarzian derivative, $S(\sigma) = \big(\sigma''/\sigma'\big)' - \frac{1}{2}\big(\sigma''/\sigma'\big)^2$.

$Diff(S^1)$ has one nontrivial coadjoint 1-cocycle. The central extension $\hat{\mathfrak{d}} = \mathfrak{d} + \mathbb{R}$ of the Lie algebra $\mathfrak{d} = Vect(S^1)$ of $Diff(S^1)$ is known as the Virasoro Lie algebra. From now on we use $\hat{\mathfrak{d}} = VIR$. The formulae for the adjoint and coadjoint actions respectively on VIR and on the dual of VIR are given by

$$\hat{\mathrm{ad}}^{VIR}_{(\phi_1,a_1,)}(\phi_2,a_2) = \Big(\phi_1\phi_2' - \phi_1'\phi_2, \int_{S^1}\phi_1\phi_2'''\Big),$$
$$(\phi_i, a_i) \in C^\infty(S^1,\mathbb{R}) \times \mathbb{R} \sim VIR; \qquad (1.6)$$

$$\hat{\mathrm{ad}}^{VIR}{}^*_{\phi}(u,e) = \big(2\phi' u + \phi u' + e\phi''', 0\big)$$
$$\phi \in C^\infty(S^1,\mathbb{R}),$$
$$(u,e) \in C^\infty(S^1,\mathbb{R}) \times \mathbb{R} \sim VIR^*; \qquad (1.7)$$

$$\hat{\mathrm{Ad}}^{VIR}{}^*_{\sigma}(u,e) = \big(\sigma'^2 u\circ\sigma + eS(\sigma), e\big),$$
$$\sigma \in Diff(S^1),$$
$$(u,e) \in C^\infty(S^1,\mathbb{R}) \times \mathbb{R} \sim VIR^*. \qquad (1.8)$$

Define a map $\mathbf{m} : \hat{\mathcal{L}}^* \to VIR^*$ by

$$\mathbf{m}(u,\xi,\mathbf{e}) = \big(e_1 u - \tfrac{1}{2}(\xi,\xi) - e_2(A,\xi'),\ e_1e_3 + (e_2A, e_2A)\big)\ . \qquad (1.9)$$

The two principal results of Ref. [2] were the following:

(i) $$\mathbf{m}\circ\hat{\mathrm{Ad}}^*_{(\sigma,g)} = \hat{\mathrm{Ad}}^{VIR}{}^*_{\sigma}\circ\mathbf{m}, \qquad (1.10)$$

where $\hat{\mathrm{Ad}}^*$ is the coadjoint action of $\mathcal{G}$ on $\hat{\mathcal{L}}^*$ given by (1.5) and $\hat{\mathrm{Ad}}^{VIR*}$ is the coadjoint action of $Diff(S^1)$ on VIR^* given by (1.8).

(ii) The map $\mathcal{M} : \ell(\hat{\mathcal{L}}^*) \to \ell(VIR^*)$ also given by the formula (1.9) is a Poisson map from $\ell(\hat{\mathcal{L}}^*)$ with the R_q-Lie–Poisson bracket to $\ell(VIR^*)$ with the R_{qe_1}-Lie–Poisson bracket.

The meaning of the expression "R_q-Lie–Poisson bracket" is given in the Appendix, where the r-matrix approach is explained for completeness.

(i) implies that (some) elements of the set $I(\hat{\mathcal{L}}^*)$, of invariant functions on $\hat{\mathcal{L}}^*$ can be found from those on VIR^*.

(ii) is useful because in the investigation of integrable systems, it is the various R-Lie–Poisson brackets on the duals of the loop algebras which are important. Although it is an immediate consequence of (i) that $\mathbf{m}$ is a Poisson map with respect to the Lie–Poisson brackets on $\hat{\mathcal{L}}^*$ and on VIR^*, it is a nontrivial fact that the same is true of the respective R-Lie–Poisson structures on $\ell(\hat{\mathcal{L}}^*)$ and on $\ell(VIR^*)$.

At the end of Ref. [2] it was remarked that $\mathbf{m}$ looks like a dual map (in the sense of dual pairs) to another rather natural map:

Let $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{R}$ be the central extension of $\tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}} = C^\infty(S^1, \mathfrak{g})$ is the Lie algebra of $\tilde{G}$. Then $\mathbf{p} : \hat{\mathcal{L}}^* \to \hat{\mathfrak{g}}^*$ given by

$$\mathbf{p}(u, \xi, \mathbf{e}) = (\xi, e_1) \tag{1.11}$$

is a Poisson map from $\hat{\mathcal{L}}^*$ to $\hat{\mathfrak{g}}^*$. The functions which commute with $\mathbf{p}^*(C^\infty(\hat{\mathfrak{g}}^*))$ with respect to the Lie–Poisson bracket on $\hat{\mathcal{L}}^*$ are the pullbacks by $\mathbf{m}$ of functions on VIR^*.

This observation concerning the duality of $\mathbf{m}$ and $\mathbf{p}$ is not a rigourous statement in so far as $\hat{\mathcal{L}}^*$ is not a symplectic space. Nonetheless it suggests that there may be some underlying "reason" behind the existence of $\mathbf{m}$ and its properties.

The intention of the present paper is to provide this underlying structure. The paper is organised as follows:

Section 2 gives various formulae for moment maps and for their associated symplectic actions. Most of these formulae can be found with small differences, in Ref. [6]. Some of them can also be found in Ref. [7].

Section 3 provides the main result of the paper, namely

(i) $I(\hat{\mathcal{L}}^*) = \mathbf{m}^* I(VIR^*)$,

(ii) The results of [2] can be described in a natural way by applying a dual-pair argument to the left and right actions of $\tilde{G}$ on $T^*\tilde{G}_k$.

In Section 4 is identified the geometric picture underlying the construction of Ref. [2] (and thence of Refs. [3] and [4]). It is suggested that this picture is of quite a general type.

In Section 5 another special case is considered of the general picture described in Section 4. One such example is described in detail, leading to the standard Lax representation for the Clebsch system. Note that this shows that we have not simply found an alternative, new notation for Ref. [2], but that there do exist other kinds of examples of the picture described in Section 4, giving credence to the suggestion that it has the status of a general construction.

There now remains the interesting question:

Do there exist any other kinds of examples *essentially* different to those described in Section 3 and in Section 5?

2. The moment mappings for the actions of $\tilde{G}$ and of $Diff(S^1)$ on $T^*\tilde{G}_k$

This section is a summary of the results described in the first half of Ref. [6], with a few minor and unimportant changes.

Let G be a Lie group. Suppose that the Lie algebra $\mathfrak{g}$ of G has an Ad-invariant inner product $(\ ,\ )$, and use this inner product to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Let $\tilde{G}$ be the loop group of G,

$$\tilde{G} = C^\infty(S^1, G), \tag{2.1a}$$

and let $\tilde{\mathfrak{g}}$ be the loop algebra of $\mathfrak{g}$,

$$\tilde{\mathfrak{g}} = C^\infty(S^1, \mathfrak{g}). \tag{2.1b}$$

We let $\tilde{\mathfrak{g}}$ be a model for the space $\tilde{\mathfrak{g}}^*$ by defining the pairing between an element $\xi \in \tilde{\mathfrak{g}} \hookrightarrow \tilde{\mathfrak{g}}^*$ and an element $v \in \tilde{\mathfrak{g}}$ by

$$\langle \xi, v\rangle = \int_{S^1} (\xi(x), v(x))dx. \tag{2.2}$$

$T^*\tilde{G}$ has a one-parameter family of symplectic structures Ω_k given by

$$\Omega_k(g,\mu) = d\langle\mu, g^{-1}dg\rangle + k\langle dgg^{-1}, d(g'g^{-1})\rangle, \qquad (g,\mu) \in \tilde{G}\times\tilde{\mathfrak{g}}^*. \tag{2.3}$$

Here we have used the identification $T^*\tilde{G} \cong \tilde{G}\times\tilde{\mathfrak{g}}^*$. We will work in the space $\tilde{G}\times\tilde{\mathfrak{g}}^*$ from now on. To write the Poisson bracket corresponding to (2.3) in components, let $F \in C^\infty(\tilde{G})$ and $\xi \in \tilde{\mathfrak{g}}$; now let f and ϕ_ξ be given by

$$f(g,\mu) = F(g) \tag{2.4a}$$

$$\phi_\xi(g,\mu) = \langle\mu, \xi\rangle. \tag{2.4b}$$

Then we have

$$\{f_1, f_2\} = 0, \tag{2.5a}$$

$$\{f, \phi_\xi\}(g,\mu) = \left.\frac{d}{dt}\right|_{t=0} F(ge^{t\xi}), \tag{2.5b}$$

$$\{\phi_\xi, \phi_\eta\}(g,\mu) = -\langle\mu, [\xi,\eta]\rangle - k\langle\eta', \xi\rangle. \tag{2.5c}$$

Remark. The important ingredient is the fact that $\tilde{G}$ has a nontrivial coadjoint cocycle mapping $\tilde{G}$ to $\tilde{\mathfrak{g}}^*$, given by $g \mapsto g'g^{-1}$. We can construct a similar deformation of the canonical two form for any Lie group with a nontrivial cocycle:

Let H be a Lie group, with Lie algebra $\mathfrak{h}$. Suppose that $\Sigma : H \to \mathfrak{h}^*$ is a coadjoint one-cocycle, i.e. Σ satisfies $\Sigma(gh) = \Sigma(g) + \mathrm{Ad}^*_g \Sigma(h) \quad \forall g, h \in H$. Then the generalisation of formula (2.3) is given by

$$\Omega_k(g,\mu) = d\langle \mu, g^{-1}dg\rangle + k\langle d\Sigma(g), dgg^{-1}\rangle, \qquad (g,\mu) \in H \times \mathfrak{h}^*, \tag{2.3$'$}$$

while (2.5c) is replaced by

$$\{\phi_\xi, \phi_\eta\}(g,\mu) = -\langle \mu, [\xi,\eta]\rangle - k\langle \xi, \sigma(\eta)\rangle. \tag{2.5c$'$}$$

(2.4a,b) and (2.5a,b) are unchanged. In (2.5c$'$), $\sigma = d\Sigma(e)$ is the derivative at the identity of Σ, i.e. $\sigma(X) = d/dt|_{t=0}\Sigma(\exp tX)$; and $\langle\ ,\ \rangle$ is the pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$. This remark will be referred to in Section 4. Note that formulae (2.3$'$) and (2.5c$'$) can be found in Refs. [7], as can (2.7$'$) below.

Remark. The extra term in the symplectic structure Ω_k, given by (2.3) or (2.3$'$), is a magnetic term. In general a magnetic term on T^*M is a 2-form given by π^*c where $\pi : T^*M \to M$ is the canonical projection to the base, and c is a closed 2-form on M. A rather general construction using Hamiltonian reduction and described for example in Section 5.2 of Ref. [9], always leads to the addition of a magnetic term. The present situation could be thought of as arising as a special case of this construction, but while some readers might find it helpful to understand this fact, it is not necessary for a complete understanding of the results described here.

For completeness let it be said:
One might start off with the cotangent bundle $T^*\hat{G}$ of the central extension $\hat{G}$ of $\tilde{G}$, and then perform Hamiltonian reduction with respect to the natural $U(1)$ action associated with the bundle structure $p : \hat{G} \to \tilde{G}$. As $U(1)$ is abelian the isotropy subgroup of any point in the dual $u(1)^*$ is the whole of $U(1)$ and hence the reduced space $j^{-1}(k)/U(1)$ is diffeomorphic to $T^*\tilde{G}$, where $j : T^*\tilde{G} \to \mathbb{R}$ is the moment map for the $U(1)$ action. The reduced symplectic structure on $T^*\tilde{G}$ is the canonical one plus an extra term (given by (2.3)), whose geometric interpretation is that it is the curvature of a $U(1)$ connection on the bundle $p : \hat{G} \to \tilde{G}$

Let us denote by $T^*\tilde{G}_k$, the cotangent bundle $T^*\tilde{G}$ together with the (noncanonical) symplectic structure given by (2.3). ($T^*\tilde{G}_0$ is $T^*\tilde{G}$ with the canonical structure.) The left and right actions of $\tilde{G}$ on $\tilde{G}$ are lifted to $T^*\tilde{G}_k$ in a way which generalises the standard $k = 0$ case to give symplectic actions. Write these actions L and R:

$$L_h : T^*\tilde{G}_k \to T^*\tilde{G}_k, \qquad L_h(g,\mu) = (hg, \mu), \tag{2.6a}$$

$$R_{h^{-1}} : T^*\tilde{G}_k \to T^*\tilde{G}_k, \qquad R_{h^{-1}}(g,\mu) = (gh^{-1}, h\mu h^{-1} + kh'h^{-1}), \tag{2.6b}$$

for any $h \in \tilde{G}$. The correspondence $h \mapsto R_{h^{-1}}$ is used for convenience to keep the order of composition, i.e. $h_1h_2 \mapsto R_{h_2^{-1}h_1^{-1}} = R_{h_1^{-1}} \circ R_{h_2^{-1}}$. The moment maps J^L and J^R, for L and R are given by

$$J^L(g,\mu) = g\mu g^{-1} + kg'g^{-1}, \tag{2.7a}$$

$$J^R(g,\mu) = -\mu. \tag{2.7b}$$

Let us now draw attention to the fact that J^L and J^R form a "dual pair" on $T^*\tilde{G}_k$ – exactly as in the standard case, $k = 0$. That is

$$J^{R^{-1}}(J^R(g,\mu)) = L_{\tilde{G}}(g,\mu) = \{L_h(g,\mu)|h \in \tilde{G}\}, \tag{2.8a}$$

$$J^{L^{-1}}(J^L(g,\mu)) = R_{\tilde{G}}(g,\mu) = \{R_{h^{-1}}(g,\mu)|h \in \tilde{G}\}. \tag{2.8b}$$

Equivalently,

$$J^{R^*}C^\infty(\hat{\mathfrak{g}}^*) = \text{centraliser of } J^{L^*}C^\infty(\hat{\mathfrak{g}}^*) \text{ with respect to the Poisson bracket on } T^*\tilde{G}_k, \tag{2.9a}$$

$$J^{L^*}C^\infty(\hat{\mathfrak{g}}^*) = \text{centraliser of } J^{R^*}C^\infty(\hat{\mathfrak{g}}^*) \text{ with respect to the Poisson bracket on } T^*\tilde{G}_k, \tag{2.9b}$$

i.e. $\{F, \varphi \circ J^L\} = 0 \quad \forall \varphi \in C^\infty(\hat{\mathfrak{g}}^*) \Leftrightarrow F = \psi \circ J^R$ for some $\psi \in C^\infty(\hat{\mathfrak{g}}^*)$, and similarly for L and R interchanged.

Note. In the general case of the Remark following (2.5), the formulae in (2.7) are replaced by

$$J^L(g,\mu) = \mathrm{Ad}_g^*\mu + k\Sigma(g), \tag{2.7a$'$}$$

$$J^R(g,\mu) = -\mu, \tag{2.7b$'$}$$

and the dual pair property always holds.

The group of diffeomorphisms on the circle has product rule $Diff(S^1) \times Diff(S^1) \to Diff(S^1)$ given by

$$(\sigma_1, \sigma_2) \mapsto \sigma_1\sigma_2 = \sigma_2 \circ \sigma_1, \tag{2.10}$$

where $\circ$ means composition. (Note that (2.10) is not the usual convention. Compare the comment following (1.2).)

The Lie algebra of $Diff(S^1)$ is $\mathfrak{d} = Vect(S^1)$, the set of vector fields on the circle. $\mathfrak{d} \cong C^\infty(S^1, \mathbb{R})$ and the Lie bracket is given by

$$[\gamma, \rho] = \gamma\rho' - \gamma'\rho. \tag{2.11}$$

The dual space $\mathfrak{d}^*$ is modelled by $C^\infty(S^1, \mathbb{R})$ with the pairing given by

$$\langle u, \gamma \rangle = \int_{S^1} u(x)\gamma(x)dx. \tag{2.12}$$

$Diff(S^1)$ has a natural action on $\tilde{G}$ given by

$$\sigma \cdot g = g \circ \sigma. \tag{2.13}$$

This action can be lifted to give a symplectic action on $T^*\tilde{G}_k$. In fact, as in Ref. [6], we consider a more general action. Let us fix

$$\theta, \phi \in \mathfrak{g} \tag{2.14}$$

to be any two elements of $\mathfrak{g}$. Define the action of $Diff(S^1)$ on $\tilde{G}$ by

$$\sigma \cdot g = e^{\theta \log \sigma'} (g \circ \sigma) e^{-\phi \log \sigma'}. \tag{2.15}$$

Let us for convenience write

$$e^{\theta \log t} = \Theta(t), \tag{2.16a}$$

$$e^{\phi \log t} = \Phi(t). \tag{2.16b}$$

By lifting the action given by (2.15) we get the symplectic action of $Diff(S^1)$ on $T^*\tilde{G}_k$ given by

$$\begin{aligned} \sigma \cdot (g, \mu) &= L_{\Theta(\sigma')} R_{\Phi(\sigma')^{-1}} (g \circ \sigma, \sigma' \mu \circ \sigma) \\ &= (\Theta(\sigma') g \circ \sigma \Phi(\sigma')^{-1}, \sigma' \Phi(\sigma') \mu \circ \sigma \Phi(\sigma')^{-1} + k \Phi(\sigma')' \Phi(\sigma')^{-1}) \\ &= \left(\Theta(\sigma') g \circ \sigma \Phi(\sigma')^{-1}, \sigma' \Phi(\sigma') \mu \circ \sigma \Phi(\sigma')^{-1} + k \frac{\sigma''}{\sigma'} \phi \right). \end{aligned} \tag{2.17}$$

To get the infinitesimal action of $\mathfrak{d}$ on $T^*\tilde{G}_k$ we consider the action of $\sigma = e + t\gamma$ (e is the identity diffeomorphism) and differentiate with respect to t at $t = 0$. We obtain the vector field X_γ given by

$$X_\gamma(g, \mu) = l_{\gamma'\theta}(g, \mu) + r_{-\gamma'\phi}(g, \mu) + (\gamma g', (\gamma\mu)'), \tag{2.18}$$

where l and r are the infinitesimal actions corresponding to L and R respectively. The vector fields given by $l_{\gamma'\theta}$ and $r_{-\gamma'\phi}$ are the hamiltonian vector fields of the functions $\langle J^L, \gamma'\theta \rangle$ and $\langle J^R, \gamma'\phi \rangle$, respectively, i.e.,

$$X_\gamma(g, \mu) = X_{\langle J^L, \gamma'\theta \rangle} + X_{\langle J^R, \gamma'\phi \rangle}(g, \mu) + (\gamma g', (\gamma\mu)'). \tag{2.19}$$

In order to find the moment map for the action given by (2.17) then, we wish to find a hamiltonian function h for which $v = X_h$, where v is the vector field given by $v(g, \mu) = (\gamma g', (\gamma\mu)')$. Let us consider the action of v on the functions f and ϕ_ξ given by (2.4):

$$(vf)(g, \mu) = \left.\frac{d}{dt}\right|_{t=0} f(g e^{t\gamma g^{-1} g'}, \mu + t(\gamma\mu)') = \left.\frac{d}{dt}\right|_{t=0} F(g e^{t\gamma g^{-1} g'}), \tag{2.20a}$$

$$(v\phi_\xi)(g, \mu) = \left.\frac{d}{dt}\right|_{t=0} \phi_\xi(g e^{t\gamma g^{-1} g'}, \mu + t(\gamma\mu)') = \langle (\gamma\mu)', \xi \rangle = -\langle \mu, \gamma\xi' \rangle. \tag{2.20b}$$

From (2.20a) we have $h(g, \mu) = \langle \mu, \gamma g^{-1} g' \rangle + \kappa(g)$ for some function κ and from (2.20b) we get the following condition on κ:

$$\left.\frac{d}{dt}\right|_{t=0} \kappa(g e^{t\xi}) = k \langle \xi', \gamma g^{-1} g' \rangle. \tag{2.21}$$

A solution of (2.21) is given by

$$\kappa(g) = \tfrac{1}{2}k\langle g^{-1}g', \gamma g^{-1}g'\rangle. \tag{2.22}$$

It follows then that X_γ given by (2.18) can be written as a hamiltonian vector field;

$$X_\gamma(g,\mu) = X_H(g,\mu), \tag{2.23}$$

where

$$\begin{aligned} H(g,\mu) = &\langle J^L(g,\mu), \gamma'\theta\rangle + \langle J^R(g,\mu), \gamma'\phi\rangle \\ &+ \langle \mu, \gamma g^{-1}g'\rangle + \tfrac{1}{2}k\langle g^{-1}g', \gamma g^{-1}g'\rangle. \end{aligned} \tag{2.24}$$

We now notice the important fact pointed out in Ref. [6]:

$$H(g,\mu) = \langle \mathcal{J}(g,\mu), \gamma\rangle, \tag{2.25}$$

where $\mathcal{J}$ can be written

$$\mathcal{J} = -(\theta, J^{L'}) - (\phi, J^{R'}) + \frac{1}{2k}\left[(J^L, J^L) - (J^R, J^R)\right]. \tag{2.26}$$

We thus have a moment map $\mathcal{J} : T^*\tilde{G}_k \to \mathfrak{d}^*$ for the action given by (2.17), and $\mathcal{J}$ can be expressed in terms of the left and right moment maps J^L and J^R.

$\mathcal{J}$ is not equivariant. We can define an equivariant moment map $\hat{\mathcal{J}} : T^*\tilde{G}_k \to VIR^*$ by

$$\hat{\mathcal{J}} = (\mathcal{J}, k((\phi,\phi) - (\theta,\theta))). \tag{2.27}$$

Here VIR means the Virasoro Lie algebra, i.e. the central extension of $\mathfrak{d}$.

To conclude this section let us observe that

$$\sigma \cdot L_h \cdot (g,\mu) = L_{\Theta(\sigma')h\circ\sigma\Theta(\sigma')^{-1}} \cdot \sigma \cdot (g,\mu). \tag{2.28}$$

It follows that

$$(\sigma, h)\cdot(g,\mu) = L_h \cdot \sigma \cdot (g,\mu) \tag{2.29}$$

defines on $T^*\tilde{G}_k$ an action of the semidirect product group $\mathcal{G} = Diff(S^1) \ltimes \tilde{G}$, with product rule given by

$$\begin{aligned} &((\sigma_1, h_1), (\sigma_2, h_2)) \mapsto (\sigma_1, h_1)(\sigma_2, h_2) \\ &\quad = (\sigma_2 \circ \sigma_1, h_1\Theta(\sigma_1')h_2 \circ \sigma_1\Theta(\sigma_1')^{-1}). \end{aligned} \tag{2.30}$$

This action has moment map

$$(\mathcal{J}, J^L) : T^*\tilde{G}_k \to \mathcal{L}^*. \tag{2.31}$$

Here $\mathcal{L}^*$ is the dual to $\mathcal{L}$, the Lie algebra of $\mathcal{G}$. The Lie bracket on $\mathcal{L}$ is given by

$$\left[(\gamma, X)(\rho, Y)\right] = (\gamma\rho' - \gamma'\rho,\ [X,Y] + \gamma'[\theta, Y] - \rho'[\theta, X] + \gamma Y' - \rho X'). \tag{2.32}$$

To obtain an equivariant moment map we take

$$\mathfrak{S} = (\mathcal{J}, J^L, k, -k\alpha, k((\phi,\phi) - (\theta,\theta))) : T^*\tilde{G}_k \to \hat{\mathcal{L}}^*. \tag{2.33}$$

Here the Lie bracket on $\hat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ is given by

$$\begin{aligned}
&[(\gamma, X, a_1, b_1, c_1)(\rho, Y, a_2, b_2, c_2)] \\
&= \Big(\gamma\rho' - \gamma'\rho,\ [X,Y] + \gamma'[\theta,Y] - \rho'[\theta,X] + \gamma Y' - \rho X', \\
&\int_{S^1} (X(x), Y'(x))dx, \int_{S^1} \frac{1}{\alpha}(\theta, \gamma''(x)Y(x) - \rho''(x)X(x))dx, \\
&\int_{S^1} \gamma(x)\rho'''(x)dx\Big).
\end{aligned} \tag{2.34}$$

In addition $\mathfrak{S}$ has the intertwining property,

$$\mathfrak{S} \circ (L_h \cdot \sigma\cdot) = \hat{\mathrm{Ad}}^{(\hat{\mathcal{L}})^*}_{(\sigma,h)}\mathfrak{S}. \tag{2.35}$$

$\hat{\mathrm{Ad}}^{(\hat{\mathcal{L}})^*}$ here refers the coadjoint action of $\mathcal{G}$ on $\hat{\mathcal{L}}^*$. The notation here agrees with that of Ref. [2]. However $\mathcal{L}$ here is slightly more general than it was in Ref. [2]. There we had $\theta \propto I$.

Note the artificial introduction of the constant α into the formula for the Lie bracket. This is done to make it clear that we can have *three independent* cocycles, so that the image of $\mathfrak{S}$ really is the whole of $\hat{\mathcal{L}}^*$.

3. Description of the map m in terms of the moment mappings $\mathfrak{S}$, J^L and J^R

We are interested in finding the set $I(\hat{\mathcal{L}}^*)$ of Ad*-invariant functions on $\hat{\mathcal{L}}^*$. This is a subset of $C^\infty(\hat{\mathcal{L}}^*)$ characterised by the following property:
$\varphi \in I(\hat{\mathcal{L}}^*)$ implies that for any point $(u, \xi, e_1, e_2, e_3) \in \hat{\mathcal{L}}^*$,

$$\varphi(\mathrm{Ad}^*_{(\sigma,h)}(u, \xi, e_1, e_2, e_3)) = \varphi(u, \xi, e_1, e_2, e_3) \quad \forall(\sigma, h) \in \mathcal{G}. \tag{3.1}$$

As explained in Section 1, at least some functions of this type are generated by a Poisson map from $\hat{\mathcal{L}}^*$to VIR^*. That is

$$f \in I(VIR^*) \Rightarrow f \circ \mathbf{m} \in I(\hat{\mathcal{L}}^*), \tag{3.2}$$

where $\mathbf{m}$ is given by (1.9). We now reconsider the problem and show that it has a natural resolution in terms of the moment maps $\mathfrak{S}$, J^L, J^R.

Let us notice that $\hat{\mathcal{J}}$ can be written as the sum of two maps,

$$\hat{\mathcal{J}} = \hat{\mathcal{J}}^L + \hat{\mathcal{J}}^R \tag{3.3}$$

where

$$\hat{\mathcal{J}}^L = \left(-(J^{L'}, \theta) + \frac{1}{2k}(J^L, J^L), -k(\theta, \theta) \right), \tag{3.4a}$$

$$\hat{\mathcal{J}}^R = \left(-(J^{R'}, \phi) - \frac{1}{2k}(J^R, J^R), k(\phi, \phi) \right), \tag{3.4b}$$

and that the duality property (2.8), (2.9) of J^L and J^R means that $\hat{\mathcal{J}}^L$ and $\hat{\mathcal{J}}^R$ themselves separately define equivariant Poisson maps from $T^*\tilde{G}_k$ to VIR^*. (Note that it is not the case that a symplectic group action is associated with either of them - although this fact need not worry us here.)

Suppose that $\varphi \in C^\infty(\hat{\mathcal{L}}^*)$ satisfies (3.1). It follows from the intertwining property (2.35) of $\mathfrak{S}$, that $\varphi \circ \mathfrak{S}$ satisfies

$$\varphi\big(\mathfrak{S}(g, \mu)\big) = \varphi\big(\mathfrak{S}(L_h \cdot \sigma \cdot (g, \mu))\big) \quad \forall (\sigma, h) \in \mathcal{G}. \tag{3.5}$$

In particular, for $\sigma = e$ we have

$$(\varphi \circ \mathfrak{S}) \circ L_h = (\varphi \circ \mathfrak{S}). \tag{3.6}$$

Hence

$$\varphi \circ \mathfrak{S} = \psi \circ J^R \tag{3.7}$$

for some $\psi \in C^\infty(\hat{\mathfrak{g}}^*)$.

We have now from (3.3) and (3.7), using (2.33),

$$\varphi \circ (\mathcal{J}^L + \mathcal{J}^R, J^L, k, -k\alpha, k((\phi, \phi) - (\theta, \theta))) = \psi \circ J^R. \tag{3.8}$$

Indeed, we must have

$$\psi \circ J^R = f \circ \hat{\mathcal{J}}^R \tag{3.9}$$

for some $f \in C^\infty(VIR^*)$. It follows then that φ is given by

$$\varphi(u, \xi, e_1, e_2, e_3) = f\left(e_1 u - e_2(\xi', \frac{1}{\alpha}\theta) - \tfrac{1}{2}(\xi, \xi), e_1 e_3 + e_2^2(\frac{1}{\alpha}\theta, \frac{1}{\alpha}\theta) \right), \tag{3.10}$$

with $f \in C^\infty(VIR^*)$. If we now impose $h = I$ in Eq. (3.5) and use the equivariance property of $\hat{\mathcal{J}}^R : T^*\tilde{G}_k \to VIR^*$, we find that f must itself be an element of $I(VIR^*)$. Comparing now Eq. (3.10) with Eq. (1.9) of Section 1, we get the result,

$$\varphi = f \circ \mathbf{m} \quad \text{for some } f \in I(VIR^*). \tag{3.11}$$

Aside. Let us recall how to define an element of $I(VIR^*)$. Consider a point $(u, e) \in VIR^*$. Define the corresponding linear problem

$$(2e\partial^2 + u)\Psi = 0. \tag{3.12}$$

The action of $Diff(S^1)$ on Ψ given by

$$\sigma \cdot \Psi = \sigma'^{-1/2}\Psi \circ \sigma \tag{3.13}$$

generates the $\hat{\mathrm{Ad}}^{VIR}{}_*$ action,

$$\sigma \cdot (u, e) = \hat{\mathrm{Ad}}^{VIR}{}^*_\sigma(u, e). \tag{3.14}$$

It follows that the set of invariant functions evaluated at the point (u, e) are generated by the eigenvalues of the monodromy matrix of the linear problem (3.12).

As formula (3.11) is the main result of this section, let us rewrite it as a theorem:

Theorem. *The set $I(\hat{\mathcal{L}}^*)$ of* Ad^**-invariants on $\hat{\mathcal{L}}^*$ is the set of functions of the form*

$$\{\varphi = f \circ \mathbf{m} \mid f \in I(VIR^*)\}. \tag{3.15}$$

Thus $\varphi(u, \xi, \mathbf{e})$ is an eigenvalue of the monodromy matrix of the linear problem

$$\left[2\left(e_1 e_3 + e_2^2(\frac{1}{\alpha}\theta, \frac{1}{\alpha}\theta)\right)\partial^2 + \left(e_1 u - e_2(\frac{1}{\alpha}\theta, \xi') - \tfrac{1}{2}(\xi, \xi)\right)\right]\Psi = 0. \tag{3.16}$$

Now for comparison with Section 1 we just put $\theta = \alpha A$.

4. The general picture

What we have been looking at in Sections 2 and 3 is the following geometric picture: We have a representation – with a central extension – of a semidirect product Lie algebra $\mathfrak{a} \ltimes \mathfrak{b}$ by hamiltonian vector fields on a symplectic manifold $\mathcal{M}$. The corresponding moment map has two components, corresponding to the natural decomposition of the semidirect product; the first component $\mathcal{J}$ is factored through a dual pair $\{P^1, P^2\}$ of moment maps on $\mathcal{M}$, i.e., it can be expressed entirely in terms of these two maps, and the second component is itself just P^1. Suppose that $\mathcal{J}$ can be written in the form

$$\mathcal{J} = \mathcal{J}^1 + \mathcal{J}^2, \tag{4.1}$$

with $\mathcal{J}^1$ expressed wholly in terms of P^1 and $\mathcal{J}^2$ expressed wholly in terms of P^2; then the mapping $\mathbf{m} : [(\mathfrak{a} \ltimes \mathfrak{b})\hat{}\,]^* \to \hat{\mathfrak{a}}^*$ given by $(\mathcal{J}, P^1) \mapsto \mathcal{J}^2$, is a Poisson mapping. $\mathbf{m}$ has the form

$$\mathbf{m} : (\alpha, \beta) \mapsto \alpha - \mathbf{b}(\beta), \tag{4.2}$$

where $\mathcal{J}^1$ is $\mathbf{b}(P^1)$. $\mathbf{m}$ is

(i) equivariant

(ii) linear in α.

Properties (i) and (ii) are enough to guarantee that $\mathbf{m}$ defines a Poisson mapping on the corresponding duals of loop algebras with their appropriate R Lie–Poisson brackets. Proving this is an exercise in the r-matrix formalism; it is demonstrated for the case of $\hat{\mathcal{L}}$ in Ref. [2], and the general case can be proved in precisely the same way.

Whilst this geometrical picture is indeed an attractive one, it needs to be asked to what extent it is justified to see it as interesting or useful. The answer would seem to be that there should at least exist examples other than that provided by $\tilde{G}$ – as in Sections 2 and 3 – embodying the above moment map construction. In the next section it will be shown that another example does indeed exist; the loop group $\tilde{G} = C^\infty(S^1, G)$ is replaced by a finite dimensional vector space with product law given by addition of vectors. We will look at the Clebsch system from this point of view, and it will be seen that the "Miura map" given by (4.2) is just the Lax representation of the Clebsch system.

In view of the existence of nontrivial examples making use of Lie groups other than $\tilde{G}$, the above description of the Miura map may well turn out to have an appealingly broad application. It would be very interesting to discover some new examples; to do this will amount to the investigation of the class of centrally extended semidirect product Lie groups.

5. Example

Let us introduce this section with the simple observation that if V is an abelian Lie algebra then any skewsymmetric two-form on V defines a nontrivial cocycle. The subgroup G of $GL(V)$ which leaves this two-form invariant can be made to act on V in the obvious way.

Let V be a real vector space with an inner product. We choose some orthonormal basis $\{e_i\}$ to identify V with $\mathbb{R}^n$: we have $\langle \xi, \eta \rangle = \hat{\xi}^T \hat{\eta}$, where $\hat{\xi} \in \mathbb{R}^n$ is given by $\xi = \sum \xi_i e_i, \hat{\xi} = (\xi_1, \dots, \xi_n)^T$, and $\hat{\eta}$ is defined similarly. V^* is identified with V. We now immediately drop the hats in all that follows and make use of the two spaces V and $\mathbb{R}^n$ interchangeably.

Define $\sigma : V \to V$ by $\sigma(\xi) = \mathbf{J}\xi$, where $\mathbf{J} \in \operatorname{End} V$ satisfies $\mathbf{J}^T = -\mathbf{J}$. Then σ defines the centrally extended Lie algebra structure on $\hat{V} = V \oplus \mathbb{R}$ given by

$$[(\xi, a)(\eta, b)] = (0, \xi^T \mathbf{J} \eta). \tag{5.1}$$

The Lie group corresponding to the abelian Lie algebra V is V itself with the multiplication rule given by addition of vectors. The group cocycle Σ on V corresponding to σ is the same as σ. Thus we have the deformed symplectic structure Ω_k on T^*V given in terms of the canonical variables $(\mathbf{q}, \mathbf{p})$ on T^*V by

$$\begin{aligned} \{q_i, q_j\} &= 0 \qquad &\text{or}\quad \{\mathbf{q}, \mathbf{q}^T\} &= \mathbf{0}, \\ \{q_i, p_j\} &= \delta_{ij} \qquad &\text{or}\quad \{\mathbf{q}, \mathbf{p}^T\} &= \mathbf{I}, \\ \{p_i, p_j\} &= -kJ_{ij} \qquad &\text{or}\quad \{\mathbf{p}, \mathbf{p}^T\} &= -k\mathbf{J}. \end{aligned} \tag{5.2}$$

As before we refer from now on to T^*V_k to imply T^*V with the above non-canonical Poisson bracket.

Note that the $(\mathbf{q}, \mathbf{p})$ are not *canonical* in the usual sense unless $k = 0$, but rather are canonical as a choice of coordinates on the manifold T^*V.

There is an action on V of the subgroup of $GL(V)$ which leaves invariant the cocycle in (5.1). It is convenient to assume from now on that $\mathbf{J}$ is nondegenerate, which means that $\dim V$ is even and that the subgroup referred to earlier is the symplectic group,

$$G = \{g \in GL(V) |\ g^T = \mathbf{J}g^{-1}\mathbf{J}^{-1}\}. \tag{5.3}$$

The action of G on V is given by

$$g \cdot \xi = g\xi. \tag{5.4}$$

As explained in Section 4 we are to be interested in looking at semidirect product Lie groups $\mathcal{A} \ltimes \mathcal{B}$ having a cocycle. The idea then is to imitate the construction of the last section where now $\mathcal{B} = V$ and $\mathcal{A}$ is the symplectic group G given by Eq. (5.3).

The left and right moment maps are given by

$$J^L(\mathbf{q},\mathbf{p}) = \mathbf{p} + k\mathbf{J}\mathbf{q}\,, \tag{5.5a}$$

$$J^R(\mathbf{q},\mathbf{p}) = -\mathbf{p}\,, \tag{5.5b}$$

and as usual J^L and J^R are a dual pair of Poisson maps.

Note that in order to write down the formulae in (5.2) and in (5.5), we made use of (2.3′), (2.5c′) and (2.7a,b′) of Section 2.

The Lie algebra of G is

$$\mathfrak{g} = \{X \in gl(V) |\ X^T = -\mathbf{J}X\mathbf{J}^{-1}\}. \tag{5.6}$$

The dual of $\mathfrak{g}$ is identified with $gl(V)/\mathfrak{g}^\perp$. It is easy to check that $gl(V) = \mathfrak{g} + \mathfrak{g}^\perp$; hence $\mathfrak{g}^* \cong \mathfrak{g}$ with $X + \mathfrak{g}^\perp \cong P_\mathfrak{g} X$ where $P_\mathfrak{g}$ is the projection onto $\mathfrak{g}$ with respect to the decomposition $gl(V) = \mathfrak{g} + \mathfrak{g}^\perp$.

The action of G on V lifts to an action on T^*V_k given by

$$g \cdot (\mathbf{q},\mathbf{p}) = (g\mathbf{q}, (g^{-1})^T\mathbf{p}) = (g\mathbf{q}, \mathbf{J}g\mathbf{J}^{-1}\mathbf{p}). \tag{5.7}$$

The moment map $\mathcal{J} : T^*V_k \to \mathfrak{g}^*$ for this action is given by

$$\mathcal{J}(\mathbf{q},\mathbf{p}) = \mathbf{q}\mathbf{p}^T - \tfrac{1}{2}k\mathbf{q}\mathbf{q}^T\mathbf{J} + \mathfrak{g}^\perp \cong \tfrac{1}{2}\mathbf{q}\mathbf{p}^T - \tfrac{1}{2}\mathbf{J}^{-1}\mathbf{p}\mathbf{q}^T\mathbf{J} - \tfrac{1}{2}k\mathbf{q}\mathbf{q}^T\mathbf{J}. \tag{5.8}$$

We can rewrite $\mathcal{J}$ in terms of J^R and J^L:

$$\mathcal{J} = \frac{1}{2k}\mathbf{J}^{-1}(J^L(J^L)^T - J^R(J^R)^T) + \mathfrak{g}^\perp \cong \frac{1}{2k}\mathbf{J}^{-1}(J^L(J^L)^T - J^R(J^R)^T). \tag{5.9}$$

The mapping $\mathcal{J} \oplus J^L : T^*V_k \to \mathcal{L}^*$ is a moment map for the action of the semidirect product Lie group $\mathcal{G} = G \ltimes V$:

The product rule on $\mathcal{G}$ is given by

$$(g,\mathbf{v})(h,\mathbf{w}) = (gh, \mathbf{v} + g\mathbf{w}). \tag{5.10}$$

The Lie algebra $\mathcal{L}$ of $\mathcal{G}$ is $\mathfrak{g} \ltimes V$. The Lie bracket is given by

$$[(X,\xi)(Y,\eta)] = (XY - YX, X\eta - Y\xi). \tag{5.11}$$

$\mathcal{J} \oplus J^L$ is not equivariant. The equivariant mapping corresponding to $\mathcal{J} \oplus J^L$ is $\mathfrak{S}$: $T^*V_k \to \hat{\mathcal{L}}^*$,

$$\mathfrak{S} = (\mathcal{J}, J^L, k), \tag{5.12}$$

where $\hat{\mathcal{L}}$ is the central extension of $\mathcal{L}$ given by

$$[(X, \xi, a)(Y, \eta, b)] = (XY - YX,\ X\eta - Y\xi,\ \xi^T \mathbf{J}\eta). \tag{5.13}$$

The extended coadjoint actions of $\mathcal{G}$ and of $\mathcal{L}$ on $\hat{\mathcal{L}}^*$ are given respectively (see (2.35)) by

$$\begin{aligned}\hat{\mathrm{Ad}}^*{}_{(g,\mathbf{v})}(\alpha, \mu, e) &= \hat{\mathfrak{S}} \circ L_{\mathbf{v}} \cdot g \cdot \hat{\mathfrak{S}}^{-1}(\alpha, \mu, e)\\ &= (g\alpha g^{-1} + \tfrac{1}{2}\mathbf{v}\mu^T g^{-1} - \tfrac{1}{2}\mathbf{J}^{-1}g^{-1T}\mu\mathbf{v}^T\mathbf{J} - \tfrac{1}{2}e\mathbf{v}\mathbf{v}^T\mathbf{J}, g^{-1T}\mu + e\mathbf{J}\mathbf{v},\ e)\\ &= (g\alpha g^{-1} + \tfrac{1}{2}\mathbf{v}\mu^T g^{-1} - \tfrac{1}{2}g\mathbf{J}^{-1}\mu\mathbf{v}^T\mathbf{J} - \tfrac{1}{2}e\mathbf{v}\mathbf{v}^T\mathbf{J}, \mathbf{J}g\mathbf{J}^{-1}\mu + e\mathbf{J}\mathbf{v},\ e).\end{aligned} \tag{5.14}$$

$$\hat{\mathrm{ad}}^*{}_{(X,\xi)}(\alpha, \mu, e) = ([X, \alpha] + \tfrac{1}{2}\xi\mu^T - \tfrac{1}{2}\mathbf{J}^{-1}\mu\xi^T\mathbf{J},\ -X^T\mu + e\mathbf{J}\xi,\ 0). \tag{5.15}$$

Suppose that we would like to use the r-matrix method to describe systems of commuting flows on $\hat{\mathcal{L}}^* \otimes \mathbb{C}[\lambda, \lambda^{-1}]$. We need to find the set of $\hat{\mathrm{Ad}}^*$-invariant functions $I(\hat{\mathcal{L}}^*)$:

Let $\mathcal{F} \in I(\hat{\mathcal{L}}^*)$. Then $\mathcal{F}(\hat{\mathrm{Ad}}^*{}_{(g,\mathbf{v})}(\alpha, \mu, e)) = \mathcal{F}(\alpha, \mu, e) \quad \forall (g, \mathbf{v}) \in \mathcal{G}$. It follows from (5.14) that $\mathcal{F} \circ \mathfrak{S} \in C^\infty(T^*V_e)$ satisfies

$$(\mathcal{F} \circ \mathfrak{S})(\mathbf{q}, \mathbf{p}) = (\mathcal{F} \circ \mathfrak{S})[L_{\mathbf{v}} \cdot g \cdot (\mathbf{q}, \mathbf{p})] \quad \forall (g, \mathbf{v}) \in \mathcal{G}, \tag{5.16}$$

$\therefore \mathcal{F} \circ \mathfrak{S} = F \circ J^R$ for some $F \in C^\infty(V^*)$ and indeed $F \circ J^R = f((1/2e)\mathbf{J}^{-1}J^R(J^R)^T)$ for some $f \in C^\infty(\mathfrak{g}^*)$; this follows from putting g = identity in (5.16); but putting $\mathbf{v} = 0$ in (5.16), we find that f must be in $I(\mathfrak{g}^*)$. Thus

$$\mathfrak{S} \circ \mathcal{F} = f\left(\frac{1}{2e}\mathbf{J}^{-1}J^R(J^R)^T\right), \quad \text{with} f \in I(\mathfrak{g}^*). \tag{5.17}$$

The appropriate "Miura map" $\mathbf{m} : \hat{\mathcal{L}}^* \to \mathfrak{g}^*$ then, is given by

$$\mathbf{m}(\alpha, \mu, e) = \alpha - \frac{1}{2e}\mathbf{J}^{-1}\mu\mu^T \tag{5.18}$$

and $I(\hat{\mathcal{L}}^*)$ is the same as $\mathbf{m}^* I(\mathfrak{g}^*)$. $I(\mathfrak{g}^*)$ is generated by functions of the form,

$$\beta \longmapsto \frac{1}{j}\operatorname{tr}\beta^j. \tag{5.19}$$

The Clebsch case of a rigid body in an ideal fluid

The Clebsch case of a rigid body in an ideal fluid fits nicely into this framework. What we find is that the analogue of the Miura map given by the formula in (5.18) is the Lax representation of the system. In fact this Lax pair was given by Perelomov in Ref. [8]. A detailed description of the use of the r-matrix approach to obtain this

Lax pair can be found in Ref. [9]. The new thing is the fact that in the present context the Clebsch system can be seen as one of the very simplest examples embodying the structures of the various systems described in Refs. [3] and [4].

Let $V = \mathbb{C}^N$. (There is no difference for present purposes between $\mathbb{R}^{2N}$ and $\mathbb{C}^N$. It is just a notational convenience for us to choose this representation.) Define the inner product on V by

$$\langle \xi, \eta \rangle = \mathrm{Re}\, \xi^\dagger \eta, \quad \xi, \eta \in \mathbb{C}^N, \tag{5.20}$$

where $\dagger$ denotes complex conjugate transpose. Let

$$\mathbf{J} = i\mathbf{I}, \tag{5.21}$$

where $\mathbf{I}$ is the identity matrix.

Let

$$G = \{g \in GL_N\mathbb{C} \mid g^\dagger = \mathbf{J} g^{-1} \mathbf{J}^{-1} = g^{-1}\} = SU(N). \tag{5.22}$$

$\mathcal{G} = G \ltimes V$ has product rule

$$(g, \mathbf{v})(h, \mathbf{w}) = (gh, \mathbf{v} + g\mathbf{w}). \tag{5.23}$$

Introduce the nondegenerate inner product on the Lie algebra $\mathcal{L}$ of $\mathcal{G}$:

$$\langle (\alpha, \mu)(X, \xi) \rangle = \mathrm{Re}[\mathrm{tr}\, \alpha X + \mu^\dagger \xi], \quad (\alpha, \mu) \in \mathcal{L}, (X, \xi) \in \mathcal{L}, \tag{5.24}$$

to identify $\mathcal{L}^* \sim \mathcal{L}$. The central extension of $\mathcal{L}$ due to the cocycle $\mathbf{J}$ is given by

$$\begin{aligned}[(X, \xi, a)(Y, \eta, b)] = (XY - YX,\ X\eta - Y\xi, \mathrm{Re}\, i\xi^\dagger \eta),& \\ (X, \xi, a), (Y, \eta, b) \in \hat{\mathcal{L}} = \mathcal{L} \oplus \mathbb{R}.&\end{aligned} \tag{5.25}$$

The coadjoint actions of $\mathcal{G}$ and of $\mathcal{L}$ on $\hat{\mathcal{L}}^*$ are given respectively by

$$\hat{\mathrm{Ad}}^*{}_{(g,\mathbf{v})}(\alpha, \mu, e) = (g\alpha g^{-1} + \tfrac{1}{2}\mathbf{v}\mu^\dagger g^{-1} - \tfrac{1}{2} g\mu \mathbf{v}^\dagger - \tfrac{1}{2} ie\mathbf{v}\mathbf{v}^\dagger,\ X\mu + ie\mathbf{v}, e), \tag{5.26}$$

$$\hat{\mathrm{ad}}^*{}_{(X,\xi)}(\alpha, \mu, e) = (X\alpha - \alpha X + \tfrac{1}{2}\xi\mu^\dagger - \tfrac{1}{2}\mu\xi^\dagger,\ X\mu + ie\xi, 0). \tag{5.27}$$

The coadjoint invariants on $su(N) \sim su(N)^*$ are generated by functions of the form

$$f_j(\beta) = \frac{1}{j} \mathrm{Re}\, \mathrm{tr}\, \beta^j, \quad \beta \in su(N), \quad j = 2, 3, \ldots, N, \tag{5.28}$$

and the functions in $I(\hat{\mathcal{L}}^*)$ are generated by the pullbacks under $\mathbf{m}$ of these, where $\mathbf{m} : \hat{\mathcal{L}}^* \to su(N)$ is given by (5.18); thus

$$\mathbf{m}(\alpha, \mu, e) = \alpha + \frac{1}{2e} i\mu\mu^\dagger. \tag{5.29}$$

We now make use of the r-matrix construction (see Appendix) applied to a twisting of the loop algebra based on $\hat{\mathcal{L}}$: Define the involutive automorphism τ on $\hat{\mathcal{L}}$ by

$$\tau(X, \xi, a) = (X^*, \xi^*, -a), \tag{5.30}$$

where $*$ denotes complex conjugation. The untwisted loop algebra based on $\hat{\mathcal{L}}$ is

$$\ell(\hat{\mathcal{L}}) = \{\textstyle\sum_{i=r}^{s}(X_i,\xi_i,a_i)\lambda^i \mid (X_i,\xi_i,a_i) \in \hat{\mathcal{L}},\ r,s \text{ integers}\}. \tag{5.31}$$

The twisted loop algebra given by τ is

$$\begin{aligned}\ell(\hat{\mathcal{L}},\tau) &= \{(X,\xi,a) \in \ell(\hat{\mathcal{L}}) \mid (X,\xi,a)(-\lambda) = (X^*,\xi^*,-a)(\lambda)\}\\ &= \{\textstyle\sum_{i=r}^{s}(X_i,\xi_i,a_i)\lambda^i \mid a_{2n} = 0; \xi_{2n}, X_{2n}, i\xi_{2n+1}, iX_{2n+1} \text{ real } \forall n,\\ &\qquad \text{and } X_i \in su(N)\ \forall i\}.\end{aligned} \tag{5.32}$$

We identify $\ell(\hat{\mathcal{L}},\tau)^* \sim \ell(\hat{\mathcal{L}},\tau)$ by using the inner product,

$$\begin{aligned}\langle\!\langle(\alpha,\mu,e)(X,\xi,a)\rangle\!\rangle &= \langle(\alpha,\mu,e)(\lambda)(X,\xi,a)(\lambda)\rangle|_{\lambda^0},\\ &\qquad (\alpha,\mu,e),(X,\xi,a) \in \ell(\hat{\mathcal{L}},\tau).\end{aligned} \tag{5.33}$$

Let us fix A to be a constant real diagonal matrix, with all eigenvalues different. The space

$$C = \{(\lambda iA + \alpha, \pi, \tfrac{1}{2}\lambda) \mid \alpha \in so(N),\ \pi \in \mathbb{R}^N\} \tag{5.34}$$

is a Poisson subspace of $\ell(\hat{\mathcal{L}},\tau)^*$ with respect to the R_0 Lie–Poisson bracket (see Appendix). Moreover it can easily be checked that C together with the R_0 Lie–Poisson bracket is the same as $(so(N) \ltimes \mathbb{R}^N)^*$ with the ordinary Lie–Poisson bracket; this is the phase space for the Clebsch system.

We get

$$\mathbf{m}(\lambda iA + \alpha, \pi, \tfrac{1}{2}\lambda) = i\lambda A + \alpha + i\lambda^{-1}\pi\pi^T = L(\lambda) \quad \text{say}. \tag{5.35}$$

The mapping $L: so(N) \ltimes \mathbb{R}^N \to$ twisted loop algebra of $su(N)$ given by (5.35) is the well known Lax representation for the Clebsch system. It is invertible and so commuting flows on $so(N) \ltimes \mathbb{R}^N$ can be found directly in the standard way:
$H \in C^\infty(\ell(\hat{\mathcal{L}},\tau))$ given by

$$H(\alpha,\mu,e) = \sum_{j=1}^{N} \frac{1}{j+1} c_j (i\lambda)^{-j+1} \operatorname{tr} \mathbf{m}(\alpha,\mu,e)^{j+1}|_{\lambda^0}, \quad c_1,\ldots,c_N \in \mathbb{R}, \tag{5.36}$$

is an element of $I(\ell(\hat{\mathcal{L}},\tau)^*)$. If we evaluate this function at a point in C we have

$$H(\lambda iA + \alpha, \pi, \tfrac{1}{2}\lambda) = \sum_{j=1}^{N} \frac{1}{j+1} c_j (i\lambda)^{-j+1} \operatorname{tr} L(\lambda)^{j+1}|_{\lambda^0}. \tag{5.37}$$

The flow corresponding to the hamiltonian H is given by the Lax equation

$$\dot{L}(\lambda) = [i\lambda B + \omega(\alpha), L(\lambda)], \tag{5.38}$$

where B and ω are given by

$$B = \sum_{j=1}^{N} c_j A^j, \qquad \omega(\alpha) = \sum_{j=1}^{N} c_j (A^{j-1}\alpha + A^{j-2}\alpha A + \cdots + \alpha A^{j-1}). \tag{5.39}$$

Formulae (5.37) and (5.38) are equivalent to the Clebsch system. That is, (5.37) is the hamiltonian function, and the Lax equation (5.38) is equivalent to the equation of motion of the Clebsch system

6. Conclusion

We examined the canonical symmetry actions of $Diff(S^1)$ and of $\tilde{G}$ on the left and on the right on the symplectic manifold $T^*\tilde{G}_k$, whose symplectic structure is a deformation of the canonical one by the addition of a term depending on the coadjoint one-cocycle on $\tilde{G}$. We found that the forms of the moment maps for these actions are such that the existence and the form of a Poisson map $\mathbf{m} : \hat{\mathcal{L}}^* \to VIR^*$, are immediately transparent. This is a reflection of the duality property of the left and right moment maps.

We have thus been able to give a simple explanation of the results of Refs. [1] and [2], which themselves incorporated all of the results of Refs. [3] and [4].

We then considered the proposition that the above discussion is nothing but a special case of a general geometric construction *based* on a dual pair of Poisson maps. We therefore have to ask whether there exist manifolds other than $T^*\tilde{G}_k$, with other moment maps, for which the underlying argument carries through, to produce something interesting.

We have shown that other examples are possible and have described an example – that of the Clebsch case of a rigid body in an ideal fluid.

It remains to be seen whether the geometric description of the Miura map, as described in this paper, will be applicable to any other examples.

Appendix A. The use of the r-matrix to find hamiltonian systems with commuting hamiltonian flows

Any vector space whose dual space has a Lie algebra structure is itself endowed with a Poisson structure corresponding to the Lie algebra structure. This Poisson structure is called the Lie–Poisson structure:

Let $\mathfrak{a}$ be a Lie algebra with Lie bracket $[\ ,\ ]$. The dual space $\mathfrak{a}^*$ has the Lie–Poisson bracket $\{\ ,\ \} : C^\infty(\mathfrak{a}^*) \wedge C^\infty(\mathfrak{a}^*) \to C^\infty(\mathfrak{a}^*)$ given by

$$\{F, G\}(\alpha) = \alpha([d_\alpha F,\ d_\alpha G]), \qquad F, G \in C^\infty(\mathfrak{a}^*). \tag{A.1}$$

Here $d_\alpha F$ and $d_\alpha G$ are the differentials of F and G which are computed by the rule

$$\beta(d_\alpha F) = \left.\frac{d}{dt}\right|_{t=0} F(\alpha + \beta t) \quad \forall \beta \in \mathfrak{a}^*. \tag{A.2}$$

For the r-matrix construction we make use of the following very special extra ingredient:

Suppose that $\mathfrak{a}$ has two Lie brackets $[\ ,\ ]$ and $[\ ,\ ]_R$ and that there exists a linear operator $R \in \mathrm{End}(\mathfrak{a})$ such that

$$[X, Y]_R = \tfrac{1}{2}[RX, Y] + \tfrac{1}{2}[X, RY] \quad \forall X, Y \in \mathfrak{a}. \tag{A.3}$$

It follows that the dual space $\mathfrak{a}^*$ has two Lie–Poisson brackets $\{\ ,\ \}$ and $\{\ ,\ \}_R$, corresponding respectively to $[\ ,\ ]$ and to $[\ ,\ ]_R$. R is called an r-matrix.

$I(\mathfrak{a}^*)$ is defined by

$$I(\mathfrak{a}^*) = \{f \in C^\infty(\mathfrak{a}^*) \mid f(\mathrm{Ad}^*_g \alpha) = f(\alpha) \quad \forall g \in A\}. \tag{A.4}$$

Here A is the group whose Lie algebra is $(\mathfrak{a}, [\ ,\ ])$.

In case we don't want to compute with A, we can obtain an infinitesimal condition by differentiating (A.4):

$$I(\mathfrak{a}^*) = \{f \in C^\infty(\mathfrak{a}^*) \mid \alpha([d_\alpha f, X]) = 0 \quad \forall X \in \mathfrak{a}\}. \tag{A.5}$$

It is a simple exercise using the above definitions, to prove the following theorem.

Theorem (see for example Ref. [9]).

(i) Let $\varphi, \psi \in I(\mathfrak{a}^*)$. Then

$$\{\varphi, \psi\}_R = 0. \tag{A.6}$$

(ii) The hamiltonian vector field corresponding to $H \in I(\mathfrak{a}^*)$, with the R Lie–Poisson bracket is given by

$$\dot\alpha = \tfrac{1}{2}\mathrm{ad}^*_{(Rd_\alpha \varphi)}\alpha, \tag{A.7}$$

where ad means the action dual to the ad-action given by $[\ ,\ ]$.*

Important example. One almost invariably makes use of the following special case:

Suppose that $\mathfrak{a}_+$ and $\mathfrak{a}_-$ are both subalgebras of $\mathfrak{a}$ with $\mathfrak{a} = \mathfrak{a}_+ + \mathfrak{a}_-$; i.e., $\mathfrak{a}_+ \cap \mathfrak{a}_- = \{0\}$ and any element X of $\mathfrak{a}$ has a unique decomposition of the form

$$X = X_+ + X_-\ , \quad \text{where } X_\pm \in \mathfrak{a}_\pm. \tag{A.8}$$

Then $R = P_+ - P_-$, where $P_\pm$ are the projections on $\mathfrak{a}$ corresponding to this decomposition, defines an r-matrix. The r-bracket of (A.3) becomes

$$[X, Y]_R = [X_+, Y_+] - [X_-, Y_-]. \tag{A.9}$$

In this special case the Theorem is equivalent to the Adler–Kostant–Symes Theorem, although it has to be said that the r-matrix still remains an extremely useful computational tool.

Canonical r-matrices on $\ell(\mathfrak{g})$. The most common application of the r-matrix theorem is to the loop algebra $\ell(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ of polynomials in λ and λ^{-1} with coefficients in a Lie algebra $\mathfrak{g}$. (λ is the spectral parameter of Inverse Scattering Theory.) $\ell(\mathfrak{g})^*$ is identified with $\ell(\mathfrak{g}^*)$ by

$$\langle \alpha, X\rangle = [\alpha(\lambda)(X(\lambda))]|_{\lambda^0} = \sum \alpha_i(X_{-i}) , \tag{A.10}$$

where $\alpha(\lambda) = \sum \alpha_i \lambda^i$ and $X(\lambda) = \sum X_j \lambda^j$. Here the subscript λ^0 indicates the coefficient of λ^0 in the series expansion. Clearly

$$\ell(\mathfrak{g})_+ = \mathfrak{g} \otimes \mathbb{C}[\lambda] \quad \text{and} \quad \ell(\mathfrak{g})_- = \mathfrak{g} \otimes \lambda^{-1}\mathbb{C}[\lambda^{-1}] \tag{A.11}$$

are subalgebras whose intersection contains only zero, and

$$\ell(\mathfrak{g}) = \ell(\mathfrak{g})_+ \oplus \ell(\mathfrak{g})_- . \tag{A.12}$$

Hence $R = P_+ - P_-$ is an r-matrix on $\ell(\mathfrak{g})$.

The following extension to the above r-matrix gives a family of compatible R Lie–Poisson brackets on $\ell(\mathfrak{g}^*)$, see Ref. [10] and also Ref. [9]. Let $q(\lambda) \in \mathbb{C}[\lambda, \lambda^{-1}]$, then it is easy to check that R_q defined by $R_q = R \circ q$, is an r-matrix on $\ell(\mathfrak{g})$ for any q. It can be checked that the space $M_{-m,n} \subset \ell(\mathfrak{g}^*)$ given by

$$M_{-m,n} = \left\{ \sum_{i=-m}^{n} \alpha_i \lambda^i \right\} \tag{A.13}$$

is a Poisson subspace with respect to the R_q Lie–Poisson bracket on $\ell(\mathfrak{g}^*)$ as long as $m \geq -1, n \geq 0$ and $-m-1 \leq$ lowest power of λ in q, $n \geq$ highest power of λ in q.

In the case of a twisted loop algebra exactly the same arguments apply, but for the family of R_q Lie–Poisson brackets now we must have $q(\lambda) \in \mathbb{C}[\lambda^2, \lambda^{-2}]$.

To define a system with commuting flows using this method then, we take any Lie algebra $\mathfrak{g}$ and immediately restrict our attention to a Poisson subspace $M_{-m,n} \subset \ell(\mathfrak{g}^*)$. we look for $I(\mathfrak{g}^*)$ and consider the evaluation of an element of $I(\mathfrak{g}^*)$ at a generic (λ-dependent) point in $M_{-m,n}$. (In practice it is quite important that things are done in this order, as the set $I(\mathfrak{g}^*)$ is typically defined by solutions of a spectral problem - as for example in (3.16) of the Theorem in Section 3 - and so concretely they are given only by evaluation at a point of $M_{-m,n}$ as terms in an asymptotic series in λ.) The result will be a series in λ. Any term in this series will be a candidate hamiltonian.

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References

[1] I. Marshall, A Lie algebraic setting for Miura maps related to an energy-dependent linear problem, Commun. Math. Phys. 133 (1990) 509–520.

[2] I. Marshall, An r-matrix interpretation of modified systems, Physica D 70 (1994) 40–60.

[3] M. Antonowicz and A.P. Fordy, A family of completely integrable multi-hamiltonian systems, Phys. Lett. A 122 (1987) 95–99; Coupled KdV equations with multi-hamiltonian structures, Physica D 28 (1987) 345–357; Coupled KdV equations associated with a novel Schrödinger spectral problem, in: *Nonlinear Evolution Equations and Dynamical Systems (NEEDS '87)*, ed. J. Leon (World Scientific, Singapore, 1988) pp. 145–160; Coupled Harry Dym equations with multi-hamiltonian structures, J. Phys. A 21 (1988) L269–L275.

[4] B.A. Kupershmidt, Lie algebras and Korteweg–de Vries equations, Physica D 27 (1987) 294–310; Long wave equations and metrizable Lie algebras, J. Math. Phys. 29 (1988) 1990–2000; A multicomponent water wave equation, J. Phys. A: Math. Gen. 18 (1985) L1119–L1122; A coupled Korteweg–de Vries equation with dispersion, J. Phys. A: Math. Gen. 18 (1985) L571–L573; Modified Korteweg–de Vries equations on euclidean Lie algebras, Int. Jour. Mod. Phys. B 3 (1989) 853–861.

[5] Q.P. Liu and I. Marshall, Two modifications of Ito's equation, Physics Letters A 160 (1991) 155–160.

[6] J. Harnad and B.A. Kupershmidt, Symplectic geometries on $T^*\tilde{G}$, Hamiltonian group actions and integrable systems, J. Geometry and Physics 16 (1995) 168–206.

[7] P. Liberman and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987);
J. Harnad, Constrained Hamiltonian systems on Lie groups, moment map reductions and central extensions, preprint CRM (1990).

[8] A.M. Perelomov, Several remarks on the integrability of the equations of motion of a rigid body in an ideal fluid, Funct. Anal. Appl. 15 (1981) 144–146.

[9] A.G. Reyman and M.A. Semenov-Tian-Shansky, Itogi Mat. Nauk 16 (VINITI, 1987); English translation in: *Encyclopaedia of Mathematics*, Vol. 16 (Springer, Berlin, 1994).

[10] A.G. Reyman and M.A. Semenov-Tian-Shansky, Compatible Poisson structures for Lax equations: An r-matrix approach, Physics Letters A 130 (1988) 456–460.